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Baxter's Solution for the Free Energy of the Chiral Potts Model

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In honor of Baxter's sixtieth birthday, we would like to review some of his work on the free energy of the chiral Potts model. In spite of the enormous complexity and difficulty of the problem, Baxter, using functional relations was able to calculate not only the free energy, but also the interfacial tension. We here show that the integral for the free energy simplifies in the superintegrable case and is identical to his earlier results using entirely different approaches. His calculations are extended to include other regions. We also attempt to clarify some of his reasoning as several steps may be mysterious at first glance.

KEY WORDS: Chiral Potts model; transfer matrices; partition function; functional relations; free energy; interfacial tension; superintegrable model.

1. INTRODUCTION

The integrable chiral Potts model is a two-dimensional lattice model—to each site of the lattice we associate a spin which takes N different values and two "rapidity lines" cross each edge.[(1)] Here we shall consider a square lattice rotated 45° so that the rapidity lines are oriented horizontally and vertically marking the commuting diagonal transfer matrices, which also commute with Hamiltonians of certain quantum spin chains.[(2)] A recent review of the model is given in ref. 3.

The Boltzmann weights for the pair interaction between the two spins on an edge are given by

$$W_{\mathrm{pq}}(n)=\left(\frac{\mu_{\mathrm{p}}}{\mu_{\mathrm{q}}}\right)^n \prod_{j=1}^{n} \frac{y_{\mathrm{q}}-x_{\mathrm{p}}\omega^j}{y_{\mathrm{p}}-x_{\mathrm{q}}\omega^j}, \qquad \bar{W}_{\mathrm{pq}}(n)=(\mu_{\mathrm{p}}\mu_{\mathrm{q}})^n \prod_{j=1}^{n} \frac{\omega x_{\mathrm{p}}-x_{\mathrm{q}}\omega^j}{y_{\mathrm{q}}-y_{\mathrm{p}}\omega^j} \tag{1.1}$$

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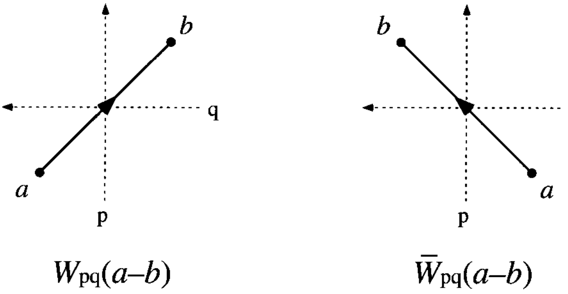

Fig. 1. Boltzmann weights $W_{\rm pq}(a-b)$ and $\bar{W}_{\rm pq}(a-b)$ for the two types of edge interaction between the spins a and b.

in which $\omega = e^{2\pi i/N}$. The weights are shown in Fig. 1, where the subscripts p and q are the two rapidity variables. We associate with every rapidity line p (or q), a variable $t_{\rm p}$ (or $t_{\rm q}$). Let $\lambda_{\rm p} = \mu_{\rm p}^N$, then λ is related to t by

$$\lambda_{\rm p} + \lambda_{\rm p}^{-1} = (1 + {\rm k}'^2 - {\rm k}^2 t_p^N)/{\rm k}' \tag{1.2}$$

where ${\rm k}^2 + {\rm k}'^2 = 1$. The variables k and k′ are fixed and the same for all the rapidity lines, and they are related to the temperature of the system (with ${\rm k}' \to 0$ for $T \to 0$ and ${\rm k}' \to 1$ for $T \to T_c$). We can uniquely determine λ from this quadratic equation by choosing the branch with either $\lambda > 1$ or $\lambda < 1$. Now, let

$$y_{\rm p}^N = (1 - {\rm k}'\lambda_{\rm p})/{\rm k}, \qquad x_{\rm p}^N = (1 - {\rm k}'/\lambda_{\rm p})/{\rm k} \tag{1.3}$$

Consequently, $x_{\rm p}$, $y_{\rm p}$ and $\mu_{\rm p}$ are given in terms of $t_{\rm p}$ up to an integral power of ω. Since $x_{\rm p}^N y_{\rm p}^N = t_{\rm p}^N$, we make the further restriction $x_{\rm p} y_{\rm p} = t_{\rm p}$. Thus the variables x, y and μ (with subscript p or q) on the right-hand side of (1.1) are now completely determined except for some irrelevant ω factors. It is easily seen from (1.3), that $\lambda \to 1/\lambda$ corresponds to interchanging x and y; thus moving from one Riemann sheet to the other corresponds to interchanging x and y in the weights.

The transfer matrices are defined by

$$T(x_{\rm q}, y_{\rm q})_{\sigma\sigma'} = \prod_{J=1}^{L} W_{\rm pq}(\sigma_J - \sigma'_J)\, \bar{W}_{\rm p'q}(\sigma_{J+1} - \sigma'_J) \tag{1.4}$$

$$\hat{T}(x_{\rm q'}, y_{\rm q'})_{\sigma'\sigma''} = \prod_{J=1}^{L} \bar{W}_{\rm pq'}(\sigma'_J - \sigma''_J)\, W_{\rm p'q'}(\sigma'_J - \sigma''_{J+1}) \tag{1.5}$$

where $L \times M$ denotes the size of the lattice. They have been shown[4] to satisfy some functional relations:

$$\Lambda_{\mathrm{q}}^{(j)} T(x_{\mathrm{q}}, y_{\mathrm{q}})\, \hat{T}(\omega^j y_{\mathrm{q}}, x_{\mathrm{q}}) = \mathbf{X}^{-j} \bar{H}_{\mathrm{p'q}}^{(j)} \tau_j(t_{\mathrm{q}}) + H_{\mathrm{pq}}^{(j)} \tau_{N-j}(\omega^j t_{\mathrm{q}}) \tag{1.6}$$

$$\tau_j(t_{\mathrm{q}})\, \tau_2(\omega^{j-1} t_{\mathrm{q}}) = \mathrm{z}(\omega^{j-1} t_{\mathrm{q}})\, \mathbf{X} \tau_{j-1}(t_{\mathrm{q}}) + \tau_{j+1}(t_{\mathrm{q}}) \tag{1.7}$$

$$\tau_{N+1}(t_{\mathrm{q}}) = \mathrm{z}(t_{\mathrm{q}})\, \mathbf{X} \tau_{N-1}(\omega t_{\mathrm{q}}) + (\alpha_{\mathrm{q}} + \bar{\alpha}_{\mathrm{q}})\, \mathbf{1} \tag{1.8}$$

where $\mathbf{X}$ is the spin shift operator,

$$\mathbf{X}_{\sigma, \sigma'} = \prod_{J=1}^{L} \delta(\sigma_J, \sigma'_J + 1), \qquad \delta(n, j) = \begin{cases} 1 & \text{if } \quad n = j \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

Unlike the eigenvalues of transfer matrices, whose dependences on t_{q} are very complicated, the elements and the eigenvalues of the matrices $\tau_j(t_{\mathrm{q}})$ are polynomials in t_{q} of degree $(j-1)\,L$ with $\tau_0(t) = 0$ and $\tau_1(t) = 1$. The scalar variables in these equations are

$$\begin{aligned} H_{\mathrm{pq}}^{(j)} &= \left[\mathrm{k} (\omega \mu_{\mathrm{p}})^j \lambda_{\mathrm{q}} \prod_{l=0}^{j-1} (t_{\mathrm{p}} - \omega^l t_{\mathrm{q}}) \right]^L \Big/ [\mathrm{k}'(1 - \lambda_{\mathrm{q}} \lambda_{\mathrm{p}})]^L \\ \bar{H}_{\mathrm{p'q}}^{(j)} &= [(1 - \lambda_{\mathrm{q}} \lambda_{\mathrm{p'}})]^L \Big/ \left[\mathrm{k} \mu_{\mathrm{p'}}^j \prod_{l=0}^{j-1} (t_{\mathrm{p'}} - \omega^l t_{\mathrm{q}}) \right]^L \end{aligned} \tag{1.10}$$

while $\Lambda_{\mathrm{q}}^{(j)} = \Lambda_{\mathrm{q}}^{(j,0)}$ with

$$\Lambda_{\mathrm{q}}^{(j)} = \left[\mu_{\mathrm{p}}^j \prod_{l=0}^{j-1} (y_{\mathrm{q}} - \omega^{-l} x_{\mathrm{p}}) \prod_{l=0}^{N-j-1} (y_{\mathrm{q}} - \omega^l y_{\mathrm{p'}}) \right]^L \Big/ [N (y_{\mathrm{p}} - x_{\mathrm{q}})(y_{\mathrm{q}} - y_{\mathrm{p'}})]^L \tag{1.11}$$

$$\mathrm{z}(t_{\mathrm{q}}) = [\omega \mu_{\mathrm{p}} \mu_{\mathrm{p'}} (t_{\mathrm{p}} - t_{\mathrm{q}})(t_{\mathrm{p'}} - t_{\mathrm{q}})]^L \tag{1.12}$$

$$\alpha_{\mathrm{q}} = [\mathrm{k}'(1 - \lambda_{\mathrm{p}} \lambda_{\mathrm{q}})(1 - \lambda_{\mathrm{p'}} \lambda_{\mathrm{q}}) / \mathrm{k}^2 \lambda_{\mathrm{q}}]^L \tag{1.13}$$

$$\bar{\alpha}_{\mathrm{q}} = [\mathrm{k}'(\lambda_{\mathrm{q}} - \lambda_{\mathrm{p}})(\lambda_{\mathrm{q}} - \lambda_{\mathrm{p'}}) / \mathrm{k}^2 \lambda_{\mathrm{q}}]^L \tag{1.14}$$

Since all the matrices commute, these relations are the functional relations between their eigenvalues. It is straightforward to verify that

$$\prod_{j=0}^{N-1} \mathrm{z}(\omega^j t_{\mathrm{q}}) = \alpha_{\mathrm{q}} \bar{\alpha}_{\mathrm{q}} \tag{1.15}$$

A few differences between Baxter's earlier paper,[5] and his later papers[6–8] are to be noted here. The normalization of weights was originally chosen[5] such that the product of the N different weights is $\prod W(n) = 1$, but it was changed to $W(0)=1$ in the later papers.[(6–8)] The latter choice is superior, as the elements of the transfer matrices are simpler, being ratios of polynomials in $x_{\rm q}$ and $y_{\rm q}$. We now list the differences,

$$
\begin{aligned}
&W_{\rm pq}(n)=W'_{\rm pq}(n)/W'_{\rm pq}(0), \qquad \rho^N_{\rm pq}=\prod_{j=0}^{N-1} W_{\rm pq}(n)=1/W'_{\rm pq}(0)^N\\
&\rho'^N_{\rm pq}=1; \qquad \tau_j(t)=(y_{\rm p}y_{\rm p'})^{(j-1)L}\,\tau'_j(t), \qquad {\rm z}(t_{\rm q})=(y_{\rm p}y_{\rm p'})^{2L}\,{\rm z}'(t_{\rm q})\\
&\alpha_{\rm q}=(y_{\rm p}y_{\rm p'})^{NL}\,\alpha'_{\rm q}, \qquad \bar\alpha_{\rm q}=(y_{\rm p}y_{\rm p'})^{NL}\,\bar\alpha'_{\rm q} \qquad (1.16)\\
&T(x_{\rm q},y_{\rm q})=[\rho_{\rm pq}\bar\rho_{\rm p'q}]^L\,T'(x_{\rm q},y_{\rm q})\\
&\hat T(x_{\rm q},y_{\rm q})=[\bar\rho_{\rm pq}\rho_{\rm p'q}]^L\,\hat T'(x_{\rm q},y_{\rm q})
\end{aligned}
$$

where the old conventions are denoted[2] with "primes." It is easily seen from (1.10) that

$$H^{(j)}_{\rm pq}/\bar H^{(j)}_{\rm p'q}=\prod_{l=0}^{j-1}{\rm z}(\omega^l t_{\rm q})/\alpha_{\rm q} \qquad (1.17)$$

Consequently, if we let

$$\Gamma^{(j)}_{\rm q}=\alpha_{\rm q}\Lambda^{(j)}_{\rm q}/\bar H^{(j)}_{\rm p'q} \qquad (1.18)$$

then (1.6) is equivalent to

$$\Gamma^{(j)}_{\rm q}\,T(x_{\rm q},y_{\rm q})\,\hat T(\omega^j y_{\rm q},x_{\rm q})={\bf X}^{-j}\alpha_{\rm q}\tau_j(t_{\rm q})+\prod_{l=0}^{j-1}z(\omega^l t_{\rm q})\,\tau_{N-j}(\omega^j t_{\rm q}) \qquad (1.19)$$

which is the form in ref. 5. It is easy to verify the following identity,

$$\Gamma^{(j)}_{\rm q}=[\rho_{\rm pq}\bar\rho_{\rm p'q}\bar\rho_{\rm pq'}\rho_{\rm p'q'}]^{-L}(y_{\rm p}y_{\rm p'})^{(N+j-1)L}\,\Gamma'^{(j)}_{\rm q} \qquad (1.20)$$

in which ${\rm q}'=\bar{\rm q}(j,0)$, a notation introduced in (BBP2.36).[3]

Iterating (1.7) $N-1$ times, and then combining with (1.8), we find

$$\tau_2(t_{\rm q})\,\tau_2(\omega t_{\rm q})\cdots\tau_2(\omega^{N-1}t_{\rm q})=(\alpha_{\rm q}+\bar\alpha_{\rm q})+\xi(t) \qquad (1.21)$$

where $\xi(t)$ is a sum of products of the polynomials $\tau_2(t)$ and ${\rm z}(t)$. From this equation all the coefficients of the polynomial $\tau_2(t)$ can be evaluated in principle by solving a system of $L-1$ coupled Nth order polynomial equations. However, as the lattice size L increases, it becomes a numerical

[2] The primes on the rapidity variables are used in both conventions, indicating that these variables may alternate in the two directions as pp′pp′ ⋯ and qq′qq′ ⋯.

[3] We shall quote equations in ref. 4 as (BBPx.xx) in the following.

nightmare. When $\tau_2(t)$ is obtained, we can use (1.7) to obtain $\tau_j(t)$ for $j=3,..., N$ successively.

It is easily verifiable[8] that

$$\begin{aligned} T(x_q, y_q)\, T(\omega x_q, y_q) \cdots T(\omega^{N-1}x_q, y_q) &= d_q S(\lambda_q) \\ \hat{T}(y_q, x_q)\, \hat{T}(\omega y_q, x_q) \cdots \hat{T}(\omega^{N-1}y_q, x_q) &= \hat{d}_q S(1/\lambda_q) \end{aligned} \tag{1.22}$$

where $S(\lambda)$ is a polynomial in λ of degree $(N-1)\,L$, and

$$\begin{aligned} 1/d_q &= c \prod_{j=1}^{N-1} (y_{p'} - \omega^j y_q)^{jL} \prod_{j=0}^{N-1} (x_p - \omega^j y_q)^{(N-1-j)\,L} \\ 1/\hat{d}_q &= c \prod_{j=1}^{N-1} (y_p - \omega^j x_q)^{jL} \prod_{j=0}^{N-1} (x_{p'} - \omega^j x_q)^{(N-1-j)\,L} \end{aligned} \tag{1.23}$$

with $c = [(\lambda_p \lambda_{p'})^{(N-1)/2}/N^N]^{L/2}$. Letting $j=0$ in (1.6), we find

$$\Lambda_q^{(0)}\, T(x_q, y_q)\, \hat{T}(y_q, x_q) = H_{pq}^{(0)} \tau_N(t_q) \tag{1.24}$$

whereas letting $j=N$ in (1.6), we obtain

$$\Lambda_q^{(N)}\, T(x_q, y_q)\, \hat{T}(y_q, x_q) = \bar{H}_{pq}^{(N)} \tau_N(t_q) \tag{1.25}$$

Thus the identity

$$\bar{H}_{pq}^{(N)}/\Lambda_q^{(N)} = H_{pq}^{(0)}/\Lambda_q^{(0)} \tag{1.26}$$

must hold, as can be easily verified.

Replacing x_q in (1.24) by $\omega x_q,..., \omega^{N-1}x_q$, and multiplying together all the N resulting equations, we get

$$S(\lambda_q)\, S(1/\lambda_q) = \mathbf{X}^{-N(N-1)/2} \tau_N(t_q)\, \tau_N(\omega t_q) \cdots \tau_N(\omega^{N-1} t_q) \tag{1.27}$$

Whenever $\tau_N(t)$ is given, this equation can be used together with (1.2) to obtain all the zeroes of $S(\lambda)$. Letting $j=0,..., N-1$ in (1.6) and multiplying these N equations together, we get

$$T(x_q, y_q)^N\, \hat{d}_q S(\lambda_q^{-1}) = \prod_{j=0}^{N-1} \{[\omega^{-j} \bar{H}_{p'q}^{(j)} \tau_j(t_q) + H_{pq}^{(j)} \tau_{N-j}(\omega^j t_q)]/\Lambda_q^{(j)}\} \tag{1.28}$$

From this equation, all the eigenvalues of the transfer matrix can be evaluated in principle when the polynomials $S(\lambda_q)$ and $\tau_j(t_q)$ are given.

The outline of this paper is as follows: We first examine the functional relations in the simplest case, namely the Ising case with $N=2$ in Section 2.

We then review, in Section 3, the steps used by Baxter to obtain the free energy for the N-state chiral Potts model for two different regimes. The results then are extended, in Section 4, to other regimes by rotations and symmetries. In Section 5 we show that the expressions simplify in the superintegrable case, and are identical to Baxter's old result.[9] There are two different but equivalent integral forms given by Baxter[5, 10, 11] for the free energy. Baxter was able to show[11] that the free energy given in the alternative integral form is equivalent to his earlier results in ref. 12 obtained using symmetries and invariances of the chiral Potts model. We simplify his procedure[5] that transforms from one form to the other in Section 6, and show that in the different regimes the alternative integral expressions are not the same, but differ slightly.

2. ISING CASE WITH $N=2$

The functional relations provide a way to calculate all the eigenvalues of the transfer matrices. We now illustrate this using the simplest case with $N=2$ which is the Ising model.

Using some special property of the Ising model and one of the functional relations, Baxter[13] was able to reproduce the result of Onsager and to obtain all the eigenvalues of the Ising model as

$$T(x_{\mathrm{q}}, y_{\mathrm{q}})=c \prod_{j\in J} (u-\mathrm{e}^{\pm i\theta_j}) \prod_{j\notin J} (\gamma\pm\gamma_j) \tag{2.1}$$

where J is some subset of the integers 1,..., $[L/2]$, and

$$u=-\sinh 2K/\sinh 2\bar{K}, \quad \gamma=\coth 2K \coth 2\bar{K} \tag{2.2}$$

Since $\mathbf{X}^2=1$, its eigenvalues are given as $r=\pm 1$. The variables θ_j in (2.1) are different for different r.

$$\theta_j=\begin{cases}(2j-1)\,\pi/L & \text{for} \quad r=1\\ 2j\pi/L & \text{for} \quad r=-1\end{cases} \tag{2.3}$$

while

$$\gamma_j=[1-2\mathrm{k}' \cos\theta_j+\mathrm{k}'^2]^{1/2} \tag{2.4}$$

By choosing different signs in (2.1), we can obtain 2×2^L different eigenvalues.

We now express these variables in terms of variables used in the chiral Potts model. They are

$$u = \frac{(t_p - t_q)(\lambda_q - \lambda_p)}{(t_p + t_q)(1 - \lambda_p \lambda_q)}, \qquad \gamma = [1 - k'(u + u^{-1}) + k'^2]^{1/2} \tag{2.5}$$

Thus, we can see from (2.1) that the eigenvalues of the transfer matrix can not be written as a product of polynomials in t and λ, different from what happens in the superintegrable case.[9]

We next illustrate how the functional relations can also be used to obtain all the eigenvalues. For $N = 2$, we replace $\mathbf{X}$ in (1.7) and (1.8) by its eigenvalue r and then combine the two equations to get

$$\tau_2(t_q)\,\tau_2(-t_q) = r[z(t_q) + z(-t_q)] + (\alpha_q + \bar{\alpha}_q) \tag{2.6}$$

where $\alpha_q \bar{\alpha}_q = z(t_q)\,z(-t_q)$. Now from (1.24), we find

$$\begin{aligned}
&\tau_2(t_q)\,\tau_2(-t_q) \\
&\quad = rS(\lambda_q)\,S(1/\lambda_q) = [\alpha_q + rz(t_q)][1 + rz(-t_q)/\alpha_q] \\
&\quad = \prod_{j=1}^{L} \{[\alpha_q^{1/L} - e^{i\theta_j}[z(t_q)]^{1/L}][1 - e^{\mp i\theta_j}[z(-t_q)/\alpha_q]^{1/L}]\} &(2.7)\\
&\quad = A \prod_{j=1}^{L} [(t_q - t_j)(t_q + t_{-j})] = B \prod_{j=1}^{L} [(\lambda_q - \lambda_j)(\lambda_q - 1/\lambda_{-j})] &(2.8)
\end{aligned}$$

In (2.7) θ_j is the same as defined in (2.3). Using (1.12)–(1.14) and (1.2), we may rewrite the product of the two factors in (2.7) as a second order polynomial in t, if the signs of θ_j in these two factors are chosen to be opposite ($\pm = -$); or as a second order polynomial in λ if the signs are chosen equal ($\pm = +$). Solving these quadratic equations, we find the roots as

$$t_j = \frac{[2ikt_p\lambda_p \sin\theta_j + (\lambda_p^2 - 1)\,\gamma_j]}{k(e^{-i\theta_j} - \lambda_p)(\lambda_p - e^{i\theta_j})}, \qquad \lambda_j = -\frac{\lambda_p e^{i\theta_j}(kt_p + \gamma_j)^2}{k'(\lambda_p - e^{i\theta_j})^2} \tag{2.9}$$

Hence,

$$\tau_2(t_q) = \sqrt{A} \prod_{j=1}^{L} (t_q \pm t_j), \qquad S(\lambda_q) = \sqrt{B} \prod_{j=1}^{L} (\lambda_q - \lambda_j^{\pm 1}) \tag{2.10}$$

Combining the two equations in (1.19) we obtain

$$\Gamma_q^{(0)} \Gamma_q^{(1)} T^2(x_q, y_q)\, S(1/\lambda_q) = \tau_2(t_q)[\alpha_q r + z(t_q)] \tag{2.11}$$

This equation is used to show that the $\pm$ signs in (2.10) are related, namely

$$\tau_2(t_{\mathrm{q}})=\sqrt{A}\prod_{j\in J}(t_{\mathrm{q}}-t_j)\prod_{j\notin J}(t_{\mathrm{q}}+t_j)$$
$$S(\lambda_{\mathrm{q}})=\sqrt{B}\prod_{j\in J}(\lambda_{\mathrm{q}}-\lambda_j)\prod_{j\notin J}(\lambda_{\mathrm{q}}-1/\lambda_j) \tag{2.12}$$

where J is some subset of integers in 1,..., L. Putting these into (2.11) we find all eigenvalues of the transfer matrix. It is highly nontrivial to show this result is identical to the one found by Baxter in (2.1), but it can be done.

3. THE LARGEST EIGENVALUE

To calculate the free energy, we need to calculate the largest eigenvalue of the transfer matrix. As in the Ising model, to determine which one is the largest eigenvalue, we need to examine the zero-temperature limit, where for the ferromagnetic case, the largest eigenvalue is known. In the chiral Potts model, the limit $T\to 0$ corresponds to $\mathrm{k}'\to 0$. It can be seen from (1.2) that, for given t, λ is either k' or $1/\mathrm{k}'$, depending on the choice of the Riemann sheet with $\lambda<1$ or $\lambda>1$. If both t_{q} and t_{p} are arbitrary, then the weights in (1.1) cannot be made to correspond to the zero-temperature ferromagnetic weights with

$$W_{\mathrm{pq}}(n)=\delta(n,0), \qquad \bar{W}_{\mathrm{pq}}(-n)=\delta(n,0) \tag{3.1}$$

If, however, we have $t^N\to 1$ for one of the rapidity variables, then the corresponding λ is finite. In this section, the case $|\mu_{\mathrm{q}}|>1$ but $|\mu_{\mathrm{p}}\mu_{\mathrm{p}'}|<1$ or $|\mu_{\mathrm{p}}\mu_{\mathrm{p}'}|>1$ will be considered. We choose

$$\lambda_{\mathrm{q}}\propto 1/\mathrm{k}', \qquad x_{\mathrm{p}}, y_{\mathrm{p}}, x_{\mathrm{p}'}, y_{\mathrm{p}'}, x_{\mathrm{q}}\to 1, \qquad y_{\mathrm{q}}\to t_{\mathrm{q}} \tag{3.2}$$

Consequently, (1.1) becomes

$$W_{\mathrm{pq}}(n)\propto \mathrm{k}'^{n/N}, \qquad \bar{W}_{\mathrm{pq}}(-n)\propto \mathrm{k}'^{n/N} \tag{3.3}$$

There are seemingly many other choices—these are the subtleties which we have not yet understood. From (3.3) we find that, as $\mathrm{k}'\to 0$, the Boltzmann weights are zero except when the adjacent spins are equal. Since the shift operator $\mathbf{X}$, the transfer matrices, and the $\tau_j(t_q)$ all commute, they can be simultaneously diagonalized. The common eigenvector, which gives the largest eigenvalue of the transfer matrix in the Q sector, is

$$|Q\rangle = \sum_{\sigma=0}^{N-1} \omega^{Q\sigma} |\sigma\rangle, \qquad |\sigma\rangle = |\sigma_1 = \sigma_2 \cdots = \sigma_L = \sigma\rangle \tag{3.4}$$

such that

$$\mathbf{X} |Q\rangle = \omega^Q |Q\rangle \tag{3.5}$$

From (BBP3.44) and (BBP3.48), for a given choice of Q, we can explicitly calculate the corresponding eigenvalue of $\tau_2(t)$ as

$$\tau_2(t_{\rm q}) = (1-\omega t_{\rm q})^L + \omega^{Q+L} (\mu_{\rm p}\mu_{\rm p'})^L (1-t_{\rm q})^L \tag{3.6}$$

From here on, we shall assume that all the matrices in the functional relations are in their diagonalized form, and we are considering now the functional relation between the leading eigenvalues whose common eigenvector gives the largest eigenvalue of the transfer matrix.

As $L \to \infty$, we find that the case $|\mu_{\rm p}\mu_{\rm p'}| > 1$ is very different from the case $|\mu_{\rm p}\mu_{\rm p'}| < 1$, namely

$$\tau_2(t_{\rm q}) = \begin{cases} (1-\omega t_{\rm q})^L & \text{for} \quad |\mu_{\rm p}\mu_{\rm p'}| < 1 \\ \omega^{Q+L}(\mu_{\rm p}\mu_{\rm p'})^L (1-t_{\rm q})^L & \text{for} \quad |\mu_{\rm p}\mu_{\rm p'}| > 1 \end{cases} \tag{3.7}$$

Consequently, for $|\mu_{\rm p}\mu_{\rm p'}| < 1$, $\tau_2(t) \sim \mathrm{O}(1)$ and its L zeroes are at ω^{-1}. As the temperature increases, we expect the L zeroes of $\tau_2(t)$ to move away but still to stay around ω^{-1}; while for $|\mu_{\rm p}\mu_{\rm p'}| > 1$, we expect $\tau_2(t) \propto (\mu_{\rm p}\mu_{\rm p'})^L$ and its L zeroes to be around 1. Now from (1.14) we find that $z(t) \propto (\mu_{\rm p}\mu_{\rm p'})^L$, thus by comparing the order of magnitude we find from (1.7) that

$$\tau_j(t) = \tau_2(t)\, \tau_2(\omega t) \cdots \tau_2(\omega^{j-2} t) \begin{cases} \propto 1 & \text{for} \ |\mu_{\rm p}\mu_{\rm p'}| < 1 \\ \propto (\mu_{\rm p}\mu_{\rm p'})^{(j-1)L} & \text{for} \ |\mu_{\rm p}\mu_{\rm p'}| > 1 \end{cases} \tag{3.8}$$

In the limit $k' \to 0$, we have $\lambda_{\rm q} \gg 1$ and

$$\alpha_{\rm q} \to [\lambda_{\rm q}\lambda_{\rm p}\lambda_{\rm p'}]^L = \lambda_{\rm q}^L [\mu_{\rm p}\mu_{\rm p'}]^{NL}, \qquad \bar{\alpha}_{\rm q} \to \lambda_{\rm q}^L \tag{3.9}$$

Using (3.8) to estimate the order of magnitude, we find in the limit $L \to \infty$, that (1.21) becomes

$$\tau_2(t_{\rm q})\, \tau_2(\omega t_{\rm q}) \cdots \tau_2(\omega^{N-1} t_{\rm q}) \to \begin{cases} \bar{\alpha}_{\rm q} & \text{if} \quad |\mu_{\rm p}\mu_{\rm p'}| < 1 \\ \alpha_{\rm q} & \text{if} \quad |\mu_{\rm p}\mu_{\rm p'}| > 1 \end{cases} \tag{3.10}$$

Thus when the right-hand side of the equation is given as a function of λ, the problem of finding $\tau_2(t)$ whose zeroes are on one of the Riemann sheets, may be viewed as a generalization of the factorization problem in Wiener–Hopf sum or integral equations. From (1.2), we write

$$t=\omega^m \hat{\Delta}(\lambda), \qquad \hat{\Delta}(\lambda)=[(1+\mathrm{k}'^2-\mathrm{k}'\lambda-\mathrm{k}'/\lambda)/\mathrm{k}^2]^{1/N} \tag{3.11}$$

such that the complex λ-plane consists of N Riemann sheets. If all the zeroes of $\tau_2(t)$ are on the lth sheet, then the $N-1$ functions $\tau_2(\omega^m t)$ for $m\neq 0$ have no zeroes on this sheet. Using Cauchy's integral formula, O'Rourke and Baxter derived that for $\lambda_q>1$, $|\mu_{\mathrm{p}}\mu_{\mathrm{p}'}|<1$ and $l=-1$ (or $N-1$)

$$\ln \tau_2(t_{\mathrm{q}})=\frac{1}{2\pi i}\oint_{|\lambda|=1} d\lambda \ln[\hat{\Delta}(\lambda)-\omega t_{\mathrm{q}}]\frac{d}{d\lambda}\ln \bar{\alpha}_{\mathrm{q}} \tag{3.12}$$

Letting

$$\lambda=\mathrm{e}^{i\theta}, \qquad \Delta(\theta)=[(1+\mathrm{k}'^2-2\mathrm{k}'\cos\theta)/\mathrm{k}^2]^{1/N}=\hat{\Delta}(\lambda) \tag{3.13}$$

the above integral can be rewritten as

$$\ln \tau_2(t_{\mathrm{q}})=\frac{L}{4\pi}\int_0^{2\pi} d\theta\left[\frac{1+\lambda_{\mathrm{p}}\mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{p}}\mathrm{e}^{i\theta}}+\frac{1+\lambda_{\mathrm{p}'}\mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{p}'}\mathrm{e}^{i\theta}}\right]\ln[\Delta(\theta)-\omega t_{\mathrm{q}}] \tag{3.14}$$

It is clearly seen from (3.11) and (3.13) that in these integrals, the functions $\Delta(\theta)$ and $\hat{\Delta}(\lambda)$ are single-valued and their arguments are in $[-\pi/N, \pi/N]$.

Similarly for $|\mu_{\mathrm{p}}\mu_{\mathrm{p}'}|>1$, when the zeroes of $\tau_2(t)$ are around 1, we find

$$\ln \tau_2(t_{\mathrm{q}})=L\ln(\omega\mu_{\mathrm{p}}\mu_{\mathrm{p}'})+\frac{1}{2\pi i}\oint_{|\lambda|=1} d\lambda \ln[\hat{\Delta}(\lambda)-t_{\mathrm{q}}]\frac{d}{d\lambda}\ln \alpha_{\mathrm{q}} \tag{3.15}$$

yielding

$$\ln\frac{\tau_2(t_{\mathrm{q}})}{(\omega\mu_{\mathrm{p}}\mu_{\mathrm{p}'})^L}=\frac{L}{4\pi}\int_0^{2\pi} d\theta\left[\frac{1+\lambda_{\mathrm{p}}^{-1}\mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{p}}^{-1}\mathrm{e}^{i\theta}}+\frac{1+\lambda_{\mathrm{p}'}^{-1}\mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{p}'}^{-1}\mathrm{e}^{i\theta}}\right]\ln[\Delta(\theta)-t_{\mathrm{q}}] \tag{3.16}$$

We may write

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\frac{1+\lambda\mathrm{e}^{i\theta}}{1-\lambda\mathrm{e}^{i\theta}}\sum_{m=0}^{N-1}\ln[\Delta(\theta)-\omega^m t]=\frac{1}{\pi}\int_0^{\pi} d\theta\,\frac{(1-\lambda^2)\ln[\Delta(\theta)^N-t^N]}{1+\lambda^2-2\lambda\cos\theta} \tag{3.17}$$

and use (1.2) and (3.13) and the integral formulae, valid for $|\lambda|$, $|\mu|<1$,

$$\int_0^\pi \frac{d\theta(1-\lambda^2)}{1+\lambda^2-2\lambda\cos\theta}=\pi, \qquad \int_0^\pi d\theta\,\frac{\ln(1+\mu^2-2\mu\cos\theta)}{1+\lambda^2-2\lambda\cos\theta}=\frac{2\pi\ln(1-\lambda\mu)}{(1-\lambda^2)} \tag{3.18}$$

to verify that the $\tau_2(t)$ given by either (3.14) or (3.16) satisfies (3.10). Even though the derivation may seem to lack rigor, and for that reason we have not even included it here, the results are indeed correct. To summarize, when the right-hand side of (3.10) is given, being a polynomial in λ related to t by (1.2), then $\tau_2(t_{\rm q})$ for $|\lambda_{\rm q}|>1$, whose zeroes are on the lth sheet of the complex λ plane, is given by

$$\ln\tau_2(t_{\rm q})=d_0+\frac{1}{2\pi i}\oint_{|\lambda|=1} d\lambda\,\ln[\omega^l\hat{\Delta}(\lambda)-t_{\rm q}]\begin{cases}\dfrac{d\ln\bar{\alpha}_{\rm q}}{d\lambda}, & \text{if}\quad |\mu_{\rm p}\mu_{\rm p'}|<1\\[2mm] \dfrac{d\ln\alpha_{\rm q}}{d\lambda}, & \text{if}\quad |\mu_{\rm p}\mu_{\rm p'}|>1\end{cases} \tag{3.19}$$

where d_0 is some constant. From (3.8), we find

$$\tau_N(t)=\tau_2(t)\,\tau_2(\omega t)\cdots\tau_2(\omega^{N-2}t) \tag{3.20}$$

Consequently, for $|\mu_{\rm p}\mu_{\rm p'}|<1$, we find that the zeroes of $\tau_N(t)$ are around $\omega^{-1}, \omega^{-2},..., \omega^{1-N}$; but not on the Riemann sheets with $t=\hat{\Delta}(\lambda)$. Therefore, we can see from (1.24), that $T(x_{\rm q}, y_{\rm q})$ cannot have zeroes on this sheet also; similarly for $|\mu_{\rm p}\mu_{\rm p'}|>1$, we find that $T(x_{\rm q}, y_{\rm q})$ has no zeroes on the Riemann sheet $t=\omega\hat{\Delta}(\lambda)$. Rewriting (1.28) as

$$T(x_{\rm q}, y_{\rm q})^N\,\hat{d}_{\rm q}\hat{S}(\lambda_{\rm q})=\lambda_{\rm q}^{(N-1)L}[H^{(0)}_{\rm pq}\tau_N(t_{\rm q})/\Lambda^{(0)}_{\rm q}]\,r(\lambda_{\rm q}, t_{\rm q}) \tag{3.21}$$

where $\hat{S}(\lambda_{\rm q})=\lambda_{\rm q}^{(N-1)L}S(1/\lambda_{\rm q})$ and

$$r(\lambda_{\rm q}, t_{\rm q})=\prod_{j=1}^{N-1}[\bar{H}^{(j)}_{\rm p'q}\tau_j(t_{\rm q})/\Lambda^{(j)}_{\rm q}]+\prod_{j=1}^{N-1}[H^{(j)}_{\rm pq}\tau_{N-j}(\omega^j t_{\rm q})/\Lambda^{(j)}_{\rm q}] \tag{3.22}$$

Baxter and O'Rourke[5, 8] then examine (3.21) for $|\mu_{\rm p}\mu_{\rm p'}|<1$, around $t_{\rm q}\sim 1$, where $T(x_{\rm q}, y_{\rm q})$ and $\tau_N(t_{\rm q})$ have no zeroes; therefore the zeroes of $r(\lambda_{\rm q}, t_{\rm q})$ are the zeroes of $\hat{S}(\lambda_{\rm q})$. Considering the limit $\mathrm{k}'\to 0$, they[8] then show that the $(N-1)\,L$ zeroes of $r(\lambda_{\rm q}, t_{\rm q})$ lie on $N-1$ circles of different radius, inside the annulus $1<|\lambda_{\rm q}|<1/\mathrm{k}'$. These zeroes can be surrounded by two contours $\mathscr{C}_-$ and $\mathscr{C}_+$. On the $N-1$ circles where the zeroes of

$r(\lambda_{\rm q}, t_{\rm q})$ are, the two terms in (3.22) must be of the same order of magnitude, but of opposite sign. As one moves away from these circles, the difference in magnitude of these two terms becomes big. Using Cauchy's integral formula, they write

$$\frac{d}{d\lambda}\ln\hat{S}(\lambda)=\frac{1}{2\pi i}\left[\oint_{\mathscr{C}_+}\frac{d\lambda'}{\lambda-\lambda'}\frac{d}{d\lambda'}\ln r(\lambda',t')-\oint_{\mathscr{C}_-}\frac{d\lambda'}{\lambda-\lambda'}\frac{d}{d\lambda'}\ln r(\lambda',t')\right] \tag{3.23}$$

in which $t'=\hat{\Delta}(\lambda')$. Guided by the results obtained in the limit $\mathrm{k}'\to 0$, Baxter and O'Rourke found that on the contour $\mathscr{C}_+$ the second term of $r(\lambda, t)$ in (3.22) dominates in the limit $L\to\infty$, and on the contour $\mathscr{C}_-$ the first term dominates. After dropping the exponentially small terms, the two contours can be shifted to the unit circle. Performing integration with respect to λ, they obtain

$$\ln\hat{S}(\lambda)=d_1+\frac{1}{2\pi i}\oint_{|\lambda'|=1}d\lambda'\ln(\lambda-\lambda')\frac{d}{d\lambda'}\sum_{j=1}^{N-1}\ln\left[\frac{H_{\rm pq}^{(j)}\tau_{N-j}(\omega^j t')}{\bar{H}_{\rm p'q}^{(j)}\tau_j(t')}\right] \tag{3.24}$$

where d_1 is some constant. We use (1.17) to find

$$\prod_{j=1}^{N-1}[H_{\rm pq}^{(j)}/\bar{H}_{\rm p'q}^{(j)}]=\alpha_{\rm q}^{-(N-1)}\prod_{l=0}^{N-1}\mathrm{z}(\omega^l t_{\rm q})^{N-1-l}=\bar{\alpha}_{\rm q}^{(N-1)}\prod_{l=1}^{N-1}\mathrm{z}(\omega^l t_{\rm q})^{-l} \tag{3.25}$$

where (1.15) is also used, and from (3.8) obtain

$$\sum_{j=1}^{N-1}\ln[\tau_{N-j}(\omega^j t')/\tau_j(t')]=\sum_{j=1}^{N-1}(N-2j)\ln\tau_2(\omega^{j-1}t') \tag{3.26}$$

Since the zeroes of $\tau_2(t)$ are around ω^{-1}, we find $\tau_2(\omega^{j-1}t)$ for $j=1,\dots,N-1$ have no zeroes on the sheet $t=\hat{\Delta}(\lambda)$, thus the above function is single-valued on this Riemann sheet. Similarly, we find $\mathrm{z}(\omega^j t)$ for $j=1,\dots,N-1$ have no zeroes on the sheet $t=\hat{\Delta}(\lambda)$ either, as seen from (1.12) and (3.2).

After substituting the second identity in (3.25) and (3.26) into (3.24), the integration involving $\bar{\alpha}_{\rm q}$ can be carried out explicitly, while the rest of the integrand has been shown to be a single-valued function on the sheet $t=\hat{\Delta}(\lambda)$. Using the identity

$$\oint d\lambda\, f(\lambda)\frac{dg(\lambda)}{d\lambda}=-\oint d\lambda\, g(\lambda)\frac{df(\lambda)}{d\lambda} \tag{3.27}$$

which is valid if $f(\lambda)$ and $g(\lambda)$ are singled-valued, and (3.14), we arrive at the final result

$$\ln \hat{S}(\lambda_q) = d_1 + (N-1)\ln \bar{\alpha}_q - \tfrac{1}{2}L[A(\lambda_q^{-1}, t_p) + A(\lambda_q^{-1}, t_{p'}) + B(\lambda_p, \lambda_q^{-1}) + B(\lambda_{p'}, \lambda_q^{-1})] \tag{3.28}$$

where

$$A(\lambda_q, t_p) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \, \frac{1+\lambda_q e^{i\theta}}{1-\lambda_q e^{i\theta}} \sum_{j=1}^{N-1} (N-j)\ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2} t_p] \tag{3.29}$$

and

$$B(\lambda_p, \lambda_q) = \frac{1}{8\pi^2}\int_0^{2\pi} d\theta \, \frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}} \int_0^{2\pi} d\phi \, \frac{1+\lambda_q e^{i\phi}}{1-\lambda_q e^{i\phi}} \times \sum_{j=1}^{N-1} (N-2j)\ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2}\Delta(\phi)] \tag{3.30}$$

If instead, we use the first identity in (3.25), we would obtain the identical result, even though it is more difficult to justify using (3.27).

These are the most crucial steps. If one uses (1.27) to determine the zeroes of $\hat{S}(\lambda)$, one would find from (3.20) that they are the image of the zeroes of $\tau_2(t)$. This means that instead of the zeroes of $\hat{S}(\lambda)$ lying on the $N-1$ circles of different radius, as implied by the solution in (3.28), they would be lying on just one circle. This just shows the ingenuity of Baxter in being able to choose the right path.

Similarly, for $|\mu_p \mu_{p'}| > 1$, we again find from (3.22) that the zeroes of $\hat{S}(\lambda)$ can be surrounded by two contours $\mathscr{C}_-$ and $\mathscr{C}_+$, and that Cauchy's integral formula (3.23) still holds. We then estimate the order of magnitude of the two terms in (3.22) for $t' \sim \omega$ in the limit $k' \to 0$. We now find on the contour $\mathscr{C}_+$ the first term in (3.22) dominating instead, while on the contour $\mathscr{C}_-$ the second term dominates. We again drop the insignificant terms, shift the two contours to the unit circle, and then integrate with respect to λ to obtain a similar equation,

$$\ln \hat{S}(\lambda) = d_2 - \frac{1}{2\pi i}\oint_{|\lambda'|=1} d\lambda' \ln(\lambda - \lambda') \frac{d}{d\lambda'} \sum_{j=1}^{N-1} \ln\left[\frac{H_{pq}^{(j)} \tau_{N-j}(\omega^j t')}{\bar{H}_{p'q}^{(j)} \tau_j(t')}\right] \tag{3.31}$$

This equation differs from (3.24) not only in the sign in front of the integral, but also in the variable t'. Here we have $t' = \omega \hat{\Delta}(\lambda')$ instead of

$t' = \hat{\Delta}(\lambda')$. It may be worthwhile to mention again, that calculating the largest eigenvalue of the transfer matrix, we find for $|\mu_{\rm p}\mu_{\rm p'}| < 1$, the zeroes of $r(\lambda_{\rm q}, t_{\rm q})$ on the Riemann sheet $t' = \hat{\Delta}(\lambda')$ are the zeroes of $\hat{S}(\lambda')$, whereas for $|\mu_{\rm p}\mu_{\rm p'}| > 1$, the zeroes of $r(\lambda_{\rm q}, t_{\rm q})$ on the sheet $t' = \omega\hat{\Delta}(\lambda')$ are the zeroes of $\hat{S}(\lambda')$. Using (3.25), (3.26) and (3.16), we find (3.31) becomes

$$\ln \hat{S}(\lambda_{\rm q}) = d_2 + (N-1) \ln \alpha_{\rm q} - \tfrac{1}{2} L[C(\lambda_{\rm q}^{-1}, t_{\rm p}) + C(\lambda_{\rm q}^{-1}, t_{\rm p'}) - B(\lambda_{\rm p}^{-1}, \lambda_{\rm q}^{-1}) - B(\lambda_{\rm p'}^{-1}, \lambda_{\rm q}^{-1})] \tag{3.32}$$

where

$$C(\lambda_{\rm q}, t_{\rm p}) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{1+\lambda_{\rm q} e^{i\theta}}{1-\lambda_{\rm q} e^{i\theta}} \sum_{j=1}^{N-1} j \ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2} t_{\rm p}] \tag{3.33}$$

Finally as $\hat{S}(\lambda)$ is now given, (3.21) can be used to calculate the largest eigenvalue of the transfer matrix, by dropping the exponentially small term in (3.22). From (1.17) and (1.14), we find that the ratio of the first term to the second term in (3.22) is of the order $(\mu_{\rm p}\mu_{\rm p'})^{N(N-1)L/2}$, thus for $|\mu_{\rm p}\mu_{\rm p'}| < 1$, the first term is exponentially small, while for $|\mu_{\rm p}\mu_{\rm p'}| > 1$, the second term is exponentially small. That is

$$\hat{S}(\lambda_{\rm q})\, T(x_{\rm q}, y_{\rm q})^N = \begin{cases} \varepsilon_{\rm q} \bar{\alpha}_{\rm q}^{(N-1)} \displaystyle\prod_{j=0}^{N-1} [\tau_{N-j}(\omega^j t_{\rm q})\, z(\omega^j t_{\rm q})^{-j}] & \text{if } |\mu_{\rm p}\mu_{\rm p'}| < 1 \\ \varepsilon_{\rm q} \displaystyle\prod_{j=1}^{N} \tau_j(t_{\rm q}) & \text{if } |\mu_{\rm p}\mu_{\rm p'}| > 1 \end{cases} \tag{3.34}$$

where

$$\varepsilon_{\rm q} = \hat{d}_{\rm q}^{-1} \lambda_{\rm q}^{(N-1)L} \prod_{j=1}^{N} [\bar{H}_{\rm p'q}^{(j)} / \Lambda_{\rm q}^{(j)}] \tag{3.35}$$

and (1.26) and (3.25) are used for the first case. It is easy to show from (1.10) and (1.11) that

$$\varepsilon_{\rm q} = [\rho_{\rm pq} \bar{D}_{\rm p'q} \Phi_0]^{NL} [\lambda_{\rm q}^2 \lambda_{\rm p}/\lambda_{\rm p'}]^{(N-1)L/4} \tag{3.36}$$

in which

$$\begin{aligned} \rho_{\rm pq}^N &= \prod_{n=1}^{N-1} W_{\rm pq}(n), & \bar{\rho}_{\rm pq}^N &= \prod_{n=1}^{N-1} \bar{W}_{\rm pq}(n) \\ D_{\rm pq}^N &= \det\nolimits_N[W_{\rm pq}(i-j)], & \bar{D}_{\rm pq}^N &= \det\nolimits_N[\bar{W}_{\rm pq}(i-j)] \end{aligned} \tag{3.37}$$

It was shown in refs. 3, 10, 12 that

$$\bar{D}^N_{\rm pq} = N^{N/2} \Phi_0^{-N} \prod_{j=1}^{N-1} \frac{(t_{\rm p} - \omega^j t_{\rm q})^j}{(y_{\rm q} - \omega^{-j} y_{\rm p})^j \, (x_{\rm p} - \omega^j x_{\rm q})^j} \tag{3.38}$$

with

$$\Phi_0 \equiv {\rm e}^{i\pi(N-1)(N-2)/12N} \tag{3.39}$$

and

$$\begin{aligned} [\bar{D}_{\rm pq}/\bar{\rho}_{\rm pq}]^N &= N^{N/2} \Phi_0^{-N} [(y^N_{\rm q} - y^N_{\rm p})(x^N_{\rm p} - x^N_{\rm q})]^{-(N-1)/2} \\ &\quad \times \prod_{j=1}^{N-1} (t_{\rm p} - \omega^j t_{\rm q})^j \\ [\bar{D}_{\rm pq} D_{\rm pq}/\rho_{\rm pq} \bar{\rho}_{\rm pq}]^N &= N^N/{\rm k}^{N-1} \end{aligned} \tag{3.40}$$

From (3.8), we find

$$\sum_{j=0}^{N-1} \ln \tau_{N-j}(\omega^j t_{\rm q}) = \sum_{j=1}^{N-1} j \ln \tau_2(\omega^{j-1} t_{\rm q}) \tag{3.41}$$

$$\sum_{j=1}^{N} \ln \tau_j(t_{\rm q}) = \sum_{j=1}^{N-1} (N-j) \ln \tau_2(\omega^{j-1} t_{\rm q}) \tag{3.42}$$

Consequently, equations (3.34) and (3.36) can be used to give the largest eigenvalue of the transfer matrix as

$$N \ln T_{\rm q} = \tfrac{1}{2} LN(\ln \tilde{\kappa}_{\rm pq} + \ln \tilde{\kappa}_{\rm p'q} + \ln \rho_{\rm pq} + \ln \bar{D}_{\rm p'q}) \tag{3.43}$$

in which we substitute (3.14) into (3.41) and use (3.28) for $\hat{S}$ to obtain, for $|\mu_{\rm p} \mu_{\rm p'}| < 1$,

$$\begin{aligned} N \ln \tilde{\kappa}_{\rm pq} &= \tfrac{1}{2}(N-1) \ln(\lambda_{\rm q}/\lambda_{\rm p}) - 2 \sum_{j=1}^{N-1} (N-j) \ln[\omega^{-j/2} t_{\rm q} - \omega^{j/2} t_{\rm p}] \\ &\quad + C(\lambda_{\rm p}, t_{\rm q}) + A(\lambda_{\rm q}^{-1}, t_{\rm p}) + B(\lambda_{\rm p}, \lambda_{\rm q}^{-1}) \end{aligned} \tag{3.44}$$

As can be seen from (3.2) and the fact that the zeroes of $\hat{S}$ are evaluated on the Riemann sheet $t_{\rm q} = \hat{\Delta}(\lambda_{\rm q})$, we find that the above expression is valid for

$$|\lambda_{\rm q}| > 1 \qquad \text{and} \qquad -\frac{\pi}{N} \leqslant \arg t_{\rm p}, \arg t_{\rm q} \leqslant \frac{\pi}{N}$$

Similarly, we use (3.16) in (3.42) and (3.32) for $\hat{S}$ to express (3.34) for $|\mu_{\rm p}\mu_{\rm p'}|>1$ as

$$N\ln\tilde{\kappa}_{\rm pq}=\tfrac{1}{2}(N-1)\ln(\lambda_{\rm q}/\lambda_{\rm p})-(N-1)\ln[{\rm k}'(1-\lambda_{\rm q}\lambda_{\rm p})^2/\lambda_{\rm q}{\rm k}^2] + A(\lambda_{\rm p}^{-1},\omega^{-1}t_{\rm q})+C(\lambda_{\rm q}^{-1},t_{\rm p})-B(\lambda_{\rm p}^{-1},\lambda_{\rm q}^{-1})+d_3 \tag{3.45}$$

where d_3 is again some constant to be determined. It is easily seen from (3.29) and (3.33) that

$$I(\lambda_{\rm p},t_{\rm q})\equiv A(\lambda_{\rm p},t_{\rm q})+C(\lambda_{\rm p},\omega t_{\rm q}) = \frac{(N-1)}{2\pi}\int_0^{2\pi}d\theta\,\frac{1+\lambda_{\rm p}{\rm e}^{i\theta}}{1-\lambda_{\rm p}{\rm e}^{i\theta}}\ln\{[\Delta(\theta)^N-t_{\rm q}^N]\,\omega^{-N^2/2}\} \tag{3.46}$$

From (3.17) and (3.18) we find the identity, for $|\lambda_{\rm p}|<1$,

$$I(\lambda_{\rm p},t_{\rm q})=-N(N-1)\frac{\pi}{2}+\begin{cases}(N-1)\ln[{\rm k}'(\lambda_{\rm q}-\lambda_{\rm p})^2/\lambda_{\rm q}{\rm k}^2] & \text{for}\quad |\lambda_{\rm q}|>1\\ (N-1)\ln[{\rm k}'(1-\lambda_{\rm p}\lambda_{\rm q})^2/\lambda_{\rm q}{\rm k}^2] & \text{for}\quad |\lambda_{\rm q}|<1\end{cases} \tag{3.47}$$

Now we use (3.47) in (3.45) to obtain

$$N\ln\tilde{\kappa}_{\rm pq}=\tfrac{1}{2}(N-1)\ln(\lambda_{\rm q}/\lambda_{\rm p})-C(\lambda_{\rm p}^{-1},t_{\rm q})+C(\lambda_{\rm q}^{-1},t_{\rm p})-B(\lambda_{\rm p}^{-1},\lambda_{\rm q}^{-1}) \tag{3.48}$$

We may also use (3.47) when p and q are interchanged to write (3.45) as

$$\ln\tilde{\kappa}_{\rm pq}^N=\tfrac{1}{2}(N-1)\ln(\lambda_{\rm p}/\lambda_{\rm q})+A(\lambda_{\rm p}^{-1},\omega^{-1}t_{\rm q})-A(\lambda_{\rm q}^{-1},\omega^{-1}t_{\rm p})-B(\lambda_{\rm p}^{-1},\lambda_{\rm q}^{-1}) \tag{3.49}$$

Since $\hat{S}$ given by (3.32) is evaluated on the Riemann sheet $t_{\rm q}=\omega\hat{\Delta}(\lambda_{\rm q})$, we find from (3.2), that equations (3.45), (3.48) and (3.49) are valid for the regime $-\pi/N\leqslant\arg t_{\rm p},\,\arg(t_{\rm q}/\omega)\leqslant\pi/N$.

When the rapidity lines satisfy ${\rm p}={\rm p}'$, the partition function is denoted by $Z_{\rm pq}$ and the partition function per site[10] is then

$$\kappa_{\rm pq}=Z_{\rm pq}^{1/ML}=\tilde{\kappa}_{\rm pq}\rho_{\rm pq}\bar{D}_{\rm pq} \tag{3.50}$$

From the inversion relation, Baxter[10] has shown

$$\tilde{\kappa}_{\rm pq}\tilde{\kappa}_{\rm qp}=1 \tag{3.51}$$

On the other hand, from (3.30), we find

$$B(\lambda_{\rm p}^{-1}, \lambda_{\rm q}^{-1}) = -B(\lambda_{\rm q}^{-1}, \lambda_{\rm p}^{-1}) \tag{3.52}$$

Consequently, we can see easily that (3.48) and (3.49) indeed satisfy this inversion relation (3.51). This shows that the constants are correctly chosen.

Baxter[5] has also shown that, for $|\mu_{\rm p}| < 1$ and $|\mu_{\rm q}| < 1$,

$$N \ln \tilde{\kappa}_{\rm pq} = \tfrac{1}{2}(N-1) \ln(\lambda_{\rm q}/\lambda_{\rm p}) + A(\lambda_{\rm p}, t_{\rm q}) - A(\lambda_{\rm q}, t_{\rm p}) - B(\lambda_{\rm p}, \lambda_{\rm q}) \tag{3.53}$$

valid in $-\pi/N \leqslant \arg t_{\rm p}, \arg t_{\rm q} \leqslant \pi/N$. It is shown in ref. 8 that (3.44) is an analytic continuation of (3.53) as $\lambda_{\rm q}$ moves from the inside of the unit circle to the outside.

4. ROTATIONS AND SYMMETRIES

The weights satisfy the properties[1, 10]

$$W_{\rm pq}(n) = \overline{W}_{\rm q^*p}(n), \qquad \overline{W}_{\rm pq}(n) = W_{\rm q^*p}(-n) \tag{4.1}$$

where $\rm q^* = R^{-1}q$, namely

$$\mu_{\rm q^*} = 1/\mu_{\rm q}, \qquad x_{\rm q^*} = \omega^{-1} y_{\rm q}, \qquad y_{\rm q^*} = x_{\rm q} \tag{4.2}$$

and

$$W_{\rm pq}(n) = \overline{W}_{\rm q,\,Rp}(-n), \qquad \overline{W}_{\rm pq}(n) = W_{\rm q,\,Rp}(-n) \tag{4.3}$$

in which

$$\mu_{\rm Rp} = 1/\mu_{\rm p}, \qquad x_{\rm Rp} = y_{\rm p}, \qquad y_{\rm Rp} = \omega x_{\rm p} \tag{4.4}$$

Combining them, we find

$$W_{\rm pq}(n) = W_{\rm Rp,\,Rq}(-n), \qquad \overline{W}_{\rm pq}(n) = \overline{W}_{\rm Rp,\,Rq}(-n) \tag{4.5}$$

From the definitions in (3.37), we obtain

$$\begin{aligned} &\rho_{\rm Rp,\,Rq} = \rho_{\rm pq}, \qquad \rho_{\rm q,\,Rp} = \bar{\rho}_{\rm pq}, \qquad \rho_{\rm q^*p} = \bar{\rho}_{\rm pq} \\ &\overline{D}_{\rm Rp,\,Rq} = \overline{D}_{\rm pq}, \qquad \overline{D}_{\rm q,\,Rp} = D_{\rm pq}, \qquad \overline{D}_{\rm q^*p} = D_{\rm pq} \end{aligned} \tag{4.6}$$

From these relations, we find that the partition per site defined in (3.50) satisfies

$$\kappa_{\mathrm{pq}} = \kappa(x_{\mathrm{p}}, y_{\mathrm{p}}, x_{\mathrm{q}}, y_{\mathrm{q}}) = \kappa_{\mathrm{Rp,\,Rq}} = \kappa(y_{\mathrm{p}}, \omega x_{\mathrm{p}}, y_{\mathrm{q}}, \omega x_{\mathrm{q}}) \tag{4.7}$$

$$= \kappa_{\mathrm{q*p}} = \kappa(\omega^{-1} y_{\mathrm{q}}, x_{\mathrm{q}}, x_{\mathrm{p}}, y_{\mathrm{p}}) \tag{4.8}$$

$$= \kappa_{\mathrm{q,\,Rp}} = \kappa(x_{\mathrm{q}}, y_{\mathrm{q}}, y_{\mathrm{p}}, \omega x_{\mathrm{p}}) \tag{4.9}$$

$$= \kappa_{\mathrm{R^2p,\,R^2q}} = \kappa(\omega x_{\mathrm{p}}, \omega y_{\mathrm{p}}, \omega x_{\mathrm{q}}, \omega y_{\mathrm{q}}) = \kappa_{\mathrm{R}^m\mathrm{p},\,\mathrm{R}^m\mathrm{q}} \tag{4.10}$$

Here,

$$\kappa_{\mathrm{R}^m\mathrm{p},\,\mathrm{R}^m\mathrm{q}} = \begin{cases} \kappa(\omega^{m/2} x_{\mathrm{p}}, \omega^{m/2} y_{\mathrm{p}}, \omega^{m/2} x_{\mathrm{q}}, \omega^{m/2} y_{\mathrm{q}}) & m \text{ even} \\ \kappa(\omega^{(m-1)/2} y_{\mathrm{p}}, \omega^{(m+1)/2} x_{\mathrm{p}}, \omega^{(m-1)/2} y_{\mathrm{q}}, \omega^{(m+1)/2} x_{\mathrm{q}}) & m \text{ odd} \end{cases} \tag{4.11}$$

As mentioned earlier, interchanging x and y is equivalent to changing λ to $1/\lambda$. Thus, these rotations allow one to extend (3.44), (3.49) and (3.53) to other regimes.

We first consider the automorphism T, given in refs. 1, 4, that leaves $t = xy$ and λ unchanged. Let

$$\mu_{\mathrm{Tq}} = \omega^{-1} \mu_{\mathrm{q}}, \qquad x_{\mathrm{Tq}} = \omega x_{\mathrm{q}}, \qquad y_{\mathrm{Tq}} = \omega^{-1} y_{\mathrm{q}} \tag{4.12}$$

Then we find from (1.1) that

$$W_{\mathrm{p,\,Tq}}(n) = \frac{W_{\mathrm{pq}}(n+1)}{W_{\mathrm{pq}}(1)}, \qquad \bar{W}_{\mathrm{p,\,Tq}}(n) = \frac{\bar{W}_{\mathrm{pq}}(n+1)}{\bar{W}_{\mathrm{pq}}(1)} \tag{4.13}$$

As a consequence, the partition function satisfies

$$Z_{\mathrm{p,\,Tq}} = [W_{\mathrm{pq}}(1)\, \bar{W}_{\mathrm{pq}}(1)]^{-ML} Z_{\mathrm{pq}} \tag{4.14}$$

From (3.37), it is seen that

$$\rho_{\mathrm{p,\,Tq}} = \frac{\rho_{\mathrm{pq}}}{W_{\mathrm{pq}}(1)}, \qquad \bar{D}_{\mathrm{p,\,Tq}} = (-1)^{(N-1)/N} \frac{\bar{D}_{\mathrm{pq}}}{\bar{W}_{\mathrm{pq}}(1)} \tag{4.15}$$

Therefore, for odd $N = 2n+1$, we find from (4.14), (4.15) and (3.50) that

$$\tilde{\kappa}_{\mathrm{p,\,Tq}} = \tilde{\kappa}(x_{\mathrm{p}}, y_{\mathrm{p}}, \omega x_{\mathrm{q}}, \omega^{-1} y_{\mathrm{q}}) = \tilde{\kappa}(x_{\mathrm{p}}, y_{\mathrm{p}}, x_{\mathrm{q}}, y_{\mathrm{q}}) \tag{4.16}$$

Similarly, we find

$$\tilde{\kappa}_{\mathrm{Tp,\,q}} = \tilde{\kappa}(\omega x_{\mathrm{p}}, \omega^{-1} y_{\mathrm{p}}, x_{\mathrm{q}}, y_{\mathrm{q}}) = \tilde{\kappa}(x_{\mathrm{p}}, y_{\mathrm{p}}, x_{\mathrm{q}}, y_{\mathrm{q}}) \tag{4.17}$$

This shows that the automorphism T leaves the normalized partition function per site $\tilde{\kappa}$ invariant for odd $N = 2n+1$.

Letting $m = 2n = N-1$ in (4.11), and using (4.6), (4.16) and (4.17), we find that (4.10) becomes

$$\tilde{\kappa}_{\mathrm{pq}} = \tilde{\kappa}(\omega^n x_{\mathrm{p}}, \omega^n y_{\mathrm{p}}, \omega^n x_{\mathrm{q}}, \omega^n y_{\mathrm{q}}) = \tilde{\kappa}(\omega^{-1} x_{\mathrm{p}}, y_{\mathrm{p}}, \omega^{-1} x_{\mathrm{q}}, y_{\mathrm{q}}) \tag{4.18}$$

in which the λ remains unchanged, but in which t_{q}, t_{p} shift to $\omega^{-1} t_{\mathrm{q}}$, $\omega^{-1} t_{\mathrm{p}}$.

Similarly, we let $m = 2n+1 = N$ in (4.11) to obtain

$$\tilde{\kappa}_{\mathrm{pq}} = \tilde{\kappa}(\omega^n y_{\mathrm{p}}, \omega^{-n} x_{\mathrm{p}}, \omega^n y_{\mathrm{q}}, \omega^{-n} x_{\mathrm{q}}) = \tilde{\kappa}(y_{\mathrm{p}}, x_{\mathrm{p}}, y_{\mathrm{q}}, x_{\mathrm{q}}) \tag{4.19}$$

Thus this transformation relates the normalized partition functions where the t_{q}, t_{p} are unchanged but λ is replaced by $1/\lambda$.

For $|\lambda_{\mathrm{p}}|, |\lambda_{\mathrm{q}}| < 1$, we find $|\lambda_{\mathrm{Rp}}|, |\lambda_{\mathrm{Rq}}| > 1$. If also $-\pi/N \leqslant \arg t_{\mathrm{Rp}} \leqslant \pi/N$, and $\pi/N \leqslant \arg t_{\mathrm{Rq}} \leqslant 3\pi/N$, then (3.49) holds for $\tilde{\kappa}_{\mathrm{Rp,\,Rq}}$. Consequently, we find using (4.6) that

$$\begin{aligned} \ln \tilde{\kappa}^N_{\mathrm{pq}} = \ln \tilde{\kappa}^N_{\mathrm{Rp,\,Rq}} &= \tfrac{1}{2}(N-1)\ln(\lambda_{\mathrm{Rp}}/\lambda_{\mathrm{Rq}}) \\ &\quad + A(\lambda_{\mathrm{Rp}}^{-1}, \omega^{-1} t_{\mathrm{Rq}}) - A(\lambda_{\mathrm{Rq}}^{-1}, \omega^{-1} t_{\mathrm{Rp}}) - B(\lambda_{\mathrm{Rp}}^{-1}, \lambda_{\mathrm{Rq}}^{-1}) \end{aligned} \tag{4.20}$$

which, as seen from (4.4), is identical to (3.53), except for the regime of validity. Combining the two regimes we find (4.24) listed in Table I and valid for $-3\pi/N \leqslant \arg(t_{\mathrm{p}}) \leqslant \pi/N$ and $-\pi/N \leqslant \arg(t_{\mathrm{q}}) \leqslant \pi/N$.

For $|\lambda_{\mathrm{p}}|, |\lambda_{\mathrm{q}}| > 1$, $-3\pi/N \leqslant \arg(t_{\mathrm{p}}) \leqslant \pi/N$ and $-\pi/N \leqslant \arg(t_{\mathrm{q}}) \leqslant \pi/N$, we use (4.19) to invert (4.24), and the result is (4.25) which is also listed in Table I, and it differs from (3.49) in that $\omega^{-1} t$ in (3.49) becomes t in (4.25). Since the regimes of validity for the two equations are different by a multiplicative ω factor, this is consistent with (4.18).

From (4.6), we find

$$\ln \tilde{\kappa}_{\mathrm{q,\,Rp}} = \ln \tilde{\kappa}_{\mathrm{q*p}} = \ln \tilde{\kappa}_{\mathrm{pq}} + \ln(\rho_{\mathrm{pq}}/\bar{\rho}_{\mathrm{pq}}) + \ln(\bar{D}_{\mathrm{pq}}/D_{\mathrm{pq}}) \tag{4.21}$$

For $|\lambda_{\mathrm{p}}| < 1$, $|\lambda_{\mathrm{q}}| > 1$, such that $|\lambda_{\mathrm{q*}}| < 1$, we consider the regime where $-3\pi/N \leqslant \arg(t_{\mathrm{q*}}) \leqslant \pi/N$ and $-\pi/N \leqslant \arg(t_{\mathrm{p}}) \leqslant \pi/N$, such that (4.24) holds for $\tilde{\kappa}_{\mathrm{q*p}}$,

$$\ln \tilde{\kappa}^N_{\mathrm{q*p}} = \tfrac{1}{2}(N-1)\ln(\lambda_{\mathrm{p}}\lambda_{\mathrm{q}}) + A(\lambda_{\mathrm{q}}^{-1}, t_{\mathrm{p}}) - A(\lambda_{\mathrm{p}}, \omega^{-1} t_{\mathrm{q}}) - B(\lambda_{\mathrm{q}}^{-1}, \lambda_{\mathrm{p}}) \tag{4.22}$$

Table I. Free Energy for Different Regions

$\lvert\lambda_{\rm p}\rvert<1,\ \lvert\lambda_{\rm q}\rvert<1$	$-\pi/N\leqslant\arg(t_{\rm q})\leqslant\pi/N$ and $-3\pi/N\leqslant\arg(t_{\rm p})\leqslant\pi/N$
$N\ln\tilde\kappa_{\rm pq}=\frac{1}{2}(N-1)\ln(\lambda_{\rm q}/\lambda_{\rm p})+A(\lambda_{\rm p},t_{\rm q})-A(\lambda_{\rm q},t_{\rm p})-B(\lambda_{\rm p},\lambda_{\rm q})$	(4.24)
$\lvert\lambda_{\rm p}\rvert>1,\ \lvert\lambda_{\rm q}\rvert>1$	$-\pi/N\leqslant\arg(t_{\rm q})\leqslant\pi/N$ and $-3\pi/N\leqslant\arg(t_{\rm p})\leqslant\pi/N$
$N\ln\tilde\kappa_{\rm pq}=\frac{1}{2}(N-1)\ln(\lambda_{\rm p}/\lambda_{\rm q})+A(\lambda_{\rm p}^{-1},t_{\rm q})-A(\lambda_{\rm q}^{-1},t_{\rm p})-B(\lambda_{\rm p}^{-1},\lambda_{\rm q}^{-1})$	(4.25)
$\lvert\lambda_{\rm p}\rvert<1,\ \lvert\lambda_{\rm q}\rvert>1$	$-\pi/N\leqslant\arg(t_{\rm q})\leqslant3\pi/N$ and $-\pi/N\leqslant\arg(t_{\rm p})\leqslant\pi/N$
$N\ln\tilde\kappa_{\rm pq}=\frac{1}{2}(N-1)\ln(\lambda_{\rm q}/\lambda_{\rm p})-2\sum_{j=1}^{N-1}(N-j)\ln(\omega^{-j/2}t_{\rm q}-\omega^{j/2}t_{\rm p})+C(\lambda_{\rm p},t_{\rm q})+A(\lambda_{\rm q}^{-1},t_{\rm p})+B(\lambda_{\rm p},\lambda_{\rm q}^{-1})$	(4.26)
$\lvert\lambda_{\rm p}\rvert>1,\ \lvert\lambda_{\rm q}\rvert<1$	$-\pi/N\leqslant\arg(t_{\rm q})\leqslant3\pi/N$ and $-\pi/N\leqslant\arg(t_{\rm p})\leqslant\pi/N$
$N\ln\tilde\kappa_{\rm pq}=\frac{1}{2}(N-1)\ln(\lambda_{\rm p}/\lambda_{\rm q})-2\sum_{j=1}^{N-1}(N-j)\ln(\omega^{-j/2}t_{\rm q}-\omega^{j/2}t_{\rm p})+C(\lambda_{\rm p}^{-1},t_{\rm q})+A(\lambda_{\rm q},t_{\rm p})+B(\lambda_{\rm p}^{-1},\lambda_{\rm q})$	(4.27)

Using (3.40) and (3.47), we may rewrite (4.21) as

$$\ln\tilde\kappa_{\rm q*p}^N=\ln\tilde\kappa_{\rm pq}^N+(N-1)\ln\lambda_{\rm p}+2\sum_{j=1}^{N-1}(N-j)\ln(\omega^{-j/2}t_{\rm q}-\omega^{j/2}t_{\rm p})$$
$$-A(\lambda_{\rm p},\omega^{-1}t_{\rm q})-C(\lambda_{\rm p},t_{\rm q})\tag{4.23}$$

Consequently, we find (4.26) in Table I, which is again identical to (3.44), but with region of validity extended. This shows that all the calculations are consistent.

Finally, for $|\lambda_{\rm p}|>1$, $|\lambda_{\rm q}|<1$, we again use (4.19) in (4.26) to obtain (4.27) which is given also in Table I. Even though equations (4.16)–(4.19) are proven here for odd N only, the results in (4.24), (4.25), (4.26) and (4.27) are valid for even N also, because we have derived these formulae using a more tedious way, namely by taking a different low-temperature $k'\to0$ limit choosing $\mu_{\rm q}\to k'$ instead of $\mu_{\rm p}\to k'$ as Baxter did in ref. 5.

The regime of $t_{\rm p}, t_{\rm q}$ for which (4.24) is valid is different from the regime for which (4.26) holds. In the intersection of these two regimes, it is found that (4.26) is an analytic continuation of (4.24) as the variable $\lambda_{\rm q}$ moves from inside the unit circle to outside the unit circle. However, since the two regimes do not coincide, it shows that this is not true in general.

Thus even though the regimes of validity for (4.27) and (4.25) do intersect, we found that (4.25) is not the analytic continuation of (4.27) when λ_p is continued from inside the unit circle to outside. When $|\lambda_p|, |\lambda_q| < 1$ or $|\lambda_p|, |\lambda_q| > 1$ we find from (4.24) or from (4.25) that the inversion relation (3.51) holds. However, if $|\lambda_p| < 1$ and $|\lambda_q| > 1$, then we need to use (4.26) for $\tilde{\kappa}_{pq}$; and (4.27) for $\tilde{\kappa}_{qp}$; we find that inversion relation (3.51) does not hold. This is rather perplexing.

5. SUPERINTEGRABLE MODEL

The model becomes superintegrable when the rapidity variables p′ and p are related according to

$$\mu_{p'} = 1/\mu_p, \qquad x_{p'} = y_p, \qquad y_{p'} = x_p \tag{5.1}$$

Consider two "column" transfer matrices associated with the two vertical rapidity lines p and p′,

$$[T_p^c]_{\sigma\sigma'} = \prod_{J=1}^{M} W_{pq}(\sigma_J - \sigma'_J)\, \bar{W}_{pq}(\sigma'_{J+1} - \sigma_J) \tag{5.2}$$

$$[\hat{T}_{p'}^c]_{\sigma'\sigma''} = \prod_{J=1}^{M} \bar{W}_{p'q}(\sigma''_J - \sigma'_J)\, W_{p'q}(\sigma'_{J+1} - \sigma''_J) \tag{5.3}$$

Using the rotation property (4.1) we turn the "column" transfer matrices into "row" transfer matrices, namely

$$[T_p^c]_{\sigma\sigma'} = \prod_{J=1}^{M} \bar{W}_{q^*p}(\sigma_J - \sigma'_J)\, W_{q^*p}(\sigma_J - \sigma'_{J+1}) \tag{5.4}$$

$$[\hat{T}_{p'}^c]_{\sigma'\sigma''} = \prod_{J=1}^{M} W_{q^*p'}(\sigma'_J - \sigma''_J)\, \bar{W}_{q^*p'}(\sigma'_{J+1} - \sigma''_J) \tag{5.5}$$

Assuming $|\lambda_p| < 1$ and $|\lambda_q| > 1$ we can use the results in (4.24) for T_p^c and (4.26) for $\hat{T}_{p'}^c$ to find

$$\begin{aligned} (N/M) \ln T_p^c &= \tfrac{1}{2}(N-1)\ln(\lambda_p\lambda_q) + \ln(\bar{\rho}_{pq}^N D_{pq}^N) \\ &\quad + A(\lambda_q^{-1}, t_p) - A(\lambda_p, \omega^{-1}t_q) - B(\lambda_q^{-1}, \lambda_p) \\ (N/M) \ln \hat{T}_{p'}^c &= \ln(\bar{\rho}_{p'q}^N D_{p'q}^N) - 2\sum_{j=1}^{N-1} j \ln[\omega^{-j/2-1}t_q - \omega^{j/2}t_p] \\ &\quad + \tfrac{1}{2}(N-1)\ln(\lambda_q/\lambda_p) \\ &\quad + C(\lambda_q^{-1}, t_p) + A(\lambda_p, \omega^{-1}t_q) + B(\lambda_q^{-1}, \lambda_p) \end{aligned} \tag{5.6}$$

Equation (3.40) can be used to obtain

$$(\bar{\rho}_{\mathrm{pq}} D_{\mathrm{pq}} \bar{\rho}_{\mathrm{p'q}} D_{\mathrm{p'q}})^N = N^N \lambda_{\mathrm{q}}^{1-N} \omega^{-N^2(N-1)/4} \prod_{j=1}^{N-1} [\omega^{-j/2-1} t_{\mathrm{q}} - \omega^{j/2} t_{\mathrm{p}}]^{2j} R^N \tag{5.7}$$

where

$$R^N = \left[\frac{(x_{\mathrm{q}} - x_{\mathrm{p}})(x_{\mathrm{q}} - y_{\mathrm{p}})}{(x_{\mathrm{q}}^N - x_{\mathrm{p}}^N)(x_{\mathrm{q}}^N - y_{\mathrm{p}}^N)} \right]^N \tag{5.8}$$

Consequently, the free energy of the superintegrable model is

$$\begin{aligned} M^{-1} \ln(T_{\mathrm{p}}^c \hat{T}_{\mathrm{p'}}^c) &= -2f/k_{\mathrm{B}} T \\ &= \ln(NR\omega^{-N(N-1)/4}) + N^{-1}[C(\lambda_{\mathrm{q}}^{-1}, t_{\mathrm{p}}) + A(\lambda_{\mathrm{q}}^{-1}, t_{\mathrm{p}})] \\ &= \ln(NR) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{1 + \lambda_{\mathrm{q}}^{-1} \mathrm{e}^{i\theta}}{1 - \lambda_{\mathrm{q}}^{-1} \mathrm{e}^{i\theta}} \ln \left[\frac{\Delta(\theta)^N - t_{\mathrm{p}}^N}{\Delta(\theta) - t_{\mathrm{p}}} \right] \end{aligned} \tag{5.9}$$

If $|\lambda_{\mathrm{q}}| < 1$, we use (4.27) for T_{p}^c and (4.25) for $\hat{T}_{\mathrm{p'}}^c$ to find the same expression except $\lambda_{\mathrm{q}}^{-1}$ is replaced by λ_{q}. In ref. 7, the free energy of the superintegrable model was given for $\lambda_{\mathrm{q}} > 1$ as

$$-\frac{2f}{k_{\mathrm{B}} T} = \ln \left[\frac{NR(t_{\mathrm{p}}^N - \eta^N)}{(t_{\mathrm{p}} - \eta)} \right] + \frac{2}{\pi} \int_\eta^{\eta^{-1}} dz \left[\frac{1}{t_{\mathrm{p}} - z} - \frac{N z^{N-1}}{t_{\mathrm{p}}^N - z^N} \right] \psi(\lambda_{\mathrm{q}}, z) \tag{5.10}$$

where

$$\eta^N = \frac{1 - \mathrm{k}'}{1 + \mathrm{k}'}, \qquad \psi(\lambda, z) = \tan^{-1} \left[\left| \frac{\lambda - 1}{\lambda + 1} \right| \left(\frac{\eta^{-N} - z^N}{z^N - \eta^N} \right)^{1/2} \right] \tag{5.11}$$

In $\psi(\lambda, z)$ the absolute sign is not necessary for $\lambda > 1$; it is inserted by us to make the above result valid for $\lambda < 1$ also. The terms inside the square bracket of the integrand can be written as derivatives with respect to z, that is

$$\begin{aligned} -\frac{2f}{k_{\mathrm{B}} T} - \ln \left[\frac{RN(t_{\mathrm{p}}^N - \eta^N)}{(t_{\mathrm{p}} - \eta)} \right] &= \frac{2}{\pi} \int_\eta^{\eta^{-1}} dz \, \frac{d}{dz} \ln \left[\frac{t_{\mathrm{p}}^N - z^N}{t_{\mathrm{p}} - z} \right] \psi(\lambda_{\mathrm{q}}, z) && (5.12) \\ &= \frac{2}{\pi} \left[\ln \left[\frac{t_{\mathrm{p}}^N - z^N}{t_{\mathrm{p}} - z} \right] \psi(\lambda_{\mathrm{q}}, z) \right] \Bigg|_\eta^{\eta^{-1}} - \frac{2}{\pi} \int_\eta^{\eta^{-1}} dz \ln \left[\frac{t_{\mathrm{p}}^N - z^N}{t_{\mathrm{p}} - z} \right] \frac{d}{dz} \psi(\lambda_{\mathrm{q}}, z) && (5.13) \\ &= -\ln \left[\frac{t_{\mathrm{p}}^N - \eta^N}{t_{\mathrm{p}} - \eta} \right] - \frac{2}{\pi} \int_0^\pi d\theta \ln \left[\frac{t_{\mathrm{p}}^N - \Delta(\theta)^N}{t_{\mathrm{p}} - \Delta(\theta)} \right] \frac{d}{d\theta} \psi(\lambda_{\mathrm{q}}, \Delta(\theta)) && (5.14) \end{aligned}$$

where we have performed integration by parts to obtain (5.13). It is easily seen from (5.11) that $\psi(\lambda_{\mathrm{q}}, \eta^{-1})=0$ and $\psi(\lambda_{\mathrm{q}}, \eta)=\pi/2$ giving a term that cancels almost all of the second term of the left-hand side of (5.12) leaving only a constant identical to the one in (5.9). Finally we can change the integration variable from z to θ according to $z=\Delta(\theta)$ as given by (3.13), and we can use

$$\psi(\lambda_{\mathrm{q}}, \Delta(\theta))=\tan^{-1} u=\frac{1}{2i} \ln \frac{(1+iu)}{(1-iu)}=\pm \ln \frac{(\lambda_{\mathrm{q}}-\mathrm{e}^{i\theta})}{(1-\lambda_{\mathrm{q}} \mathrm{e}^{i\theta})} \tag{5.15}$$

with + for $|\lambda_{\mathrm{q}}|>1$ and − for $|\lambda_{\mathrm{q}}|<1$, in (5.14) to show that the two expressions in (5.9) and (5.14) are identical.

6. ALTERNATIVE FORM

There are many different ways of writing the integrals for the free energy of the chiral Potts model. For example, Baxter has shown in ref. 11 that the free energy obtained from the functional relations is equivalent to the result of his earlier calculation in ref. 12 obtained using symmetries and invariances of the weights. However in ref. 11, he uses instead of the results in (4.24), an alternative form. For this reason, we feel that it is necessary to transform the results listed in the table to such alternative forms. The procedures used here are simpler than the ones originally used by Baxter.[5]

Rewrite (3.29), (3.33) and (3.30) as

$$A(\lambda_{\mathrm{q}}, t_{\mathrm{p}})=\frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1+\lambda_{\mathrm{q}} \mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{q}} \mathrm{e}^{i\theta}} M(\Delta(\theta), t_{\mathrm{p}}) \tag{6.1}$$

$$C(\lambda_{\mathrm{q}}, t_{\mathrm{p}})=\frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1+\lambda_{\mathrm{q}} \mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{q}} \mathrm{e}^{i\theta}} L(\Delta(\theta), t_{\mathrm{p}}) \tag{6.2}$$

$$B(\lambda_{\mathrm{p}}, \lambda_{\mathrm{q}})=\frac{1}{8\pi^2} \int_0^{2\pi} d\theta \frac{1+\lambda_{\mathrm{p}} \mathrm{e}^{i\theta}}{1-\lambda_{\mathrm{p}} \mathrm{e}^{i\theta}} \int_0^{2\pi} d\phi \frac{1+\lambda_{\mathrm{q}} \mathrm{e}^{i\phi}}{1-\lambda_{\mathrm{q}} \mathrm{e}^{i\phi}} N(\Delta(\theta), \Delta(\phi)) \tag{6.3}$$

where

$$M(\Delta(\theta), t_{\mathrm{p}})=\sum_{j=1}^{N-1} (N-j) \ln[\omega^{-j/2} \Delta(\theta)-\omega^{j/2} t_{\mathrm{p}}] \tag{6.4}$$

$$L(\Delta(\theta), t_{\mathrm{p}})=\sum_{j=1}^{N-1} j \ln[\omega^{-j/2} \Delta(\theta)-\omega^{j/2} t_{\mathrm{p}}] \tag{6.5}$$

$$N(\Delta(\theta), \Delta(\phi))=\sum_{j=1}^{N-1} (N-2j) \ln[\omega^{-j/2} \Delta(\theta)-\omega^{j/2} \Delta(\phi)] \tag{6.6}$$

It is easily seen that

$$N(t,s)=\sum_{j=1}^{N-1}(N-j)\ln[\omega^{-j/2}t-\omega^{j/2}s]-\sum_{j=1}^{N-1}j\ln[\omega^{-j/2}t-\omega^{j/2}s]$$
$$=\sum_{j=1}^{N-1}j\ln[(s-\omega^{j}t)/(t-\omega^{j}s)] \tag{6.7}$$

Now letting $t/s=\mathrm{e}^{2x/N}$ and $a=2j-N$ and using

$$\ln\left[\frac{1+\mathrm{e}^{(i\pi a+2x)/N}}{\mathrm{e}^{i\pi a/N}+\mathrm{e}^{2x/N}}\right]=\mathrm{P}\int_{-\infty}^{\infty}d\beta\,\frac{\mathrm{e}^{2i\beta x/\pi}}{\beta\,\mathrm{sh}\,N\beta}\,\mathrm{sh}\,a\beta \tag{6.8}$$

which is valid for $|\mathrm{Re}\,a|<N$ and $|\mathrm{Im}\,x|<\frac{1}{2}\pi(N-|\mathrm{Re}\,a|)$, with P denoting the principal value, we may express (6.7) as

$$N(t,s)=\frac{1}{2}\,\mathrm{P}\int_{-\infty}^{\infty}d\beta\,\frac{(t/s)^{i\beta N/\pi}}{\beta\,\mathrm{sh}\,N\beta}\,\mathsf{s}(\beta) \tag{6.9}$$

where

$$\mathsf{s}(\beta)=2\sum_{j=1}^{N-1}j\,\mathrm{sh}(2j-N)\,\beta$$
$$=\left[\frac{N\,\mathrm{ch}(N-1)\,\beta}{\mathrm{sh}\,\beta}-\frac{\mathrm{sh}\,N\beta}{\mathrm{sh}^{2}\,\beta}\right]=-\mathsf{s}(-\beta) \tag{6.10}$$

Likewise (6.5) may be put in the form

$$L(t,s)=\frac{1}{2}\sum_{j=1}^{N-1}(N-j)\ln[\omega^{-j/2}s-\omega^{j/2}t]+\frac{1}{2}\sum_{j=1}^{N-1}j\ln[\omega^{-j/2}t-\omega^{j/2}s]$$
$$=-\frac{1}{4}N(N-1)\,\pi i+\frac{1}{2}N\ln\left[\frac{s^{N}-t^{N}}{s-t}\right]-\frac{1}{2}\sum_{j=1}^{N-1}j\ln\left[\frac{s-\omega^{j}t}{t-\omega^{j}s}\right] \tag{6.11}$$

Again, if we let $t/s=\mathrm{e}^{2x/N}$ and use (6.8) and

$$\ln\left[\frac{\mathrm{sh}\,x}{\mathrm{sh}(x/N)}\right]=-\frac{1}{2}\,\mathrm{P}\int_{-\infty}^{\infty}d\beta\,\frac{\mathrm{e}^{2i\beta x/\pi}}{\beta\,\mathrm{sh}\,N\beta}\,\frac{\mathrm{sh}(N-1)\,\beta}{\mathrm{sh}\,\beta} \tag{6.12}$$

which is valid for $|\mathrm{Im}\, x| < \pi$, we can express (6.11) as

$$L(t, s) = M(s, t) = \frac{1}{4} N(N-1)[-\pi i + \ln(ts)] - \frac{1}{4} \mathrm{P} \int_{-\infty}^{\infty} d\beta \, \frac{(t/s)^{i\beta N/\pi}}{\beta \,\mathrm{sh}\, N\beta} [\mathsf{t}(\beta) + \mathsf{s}(\beta)], \qquad \mathsf{t}(\beta) = \frac{N \,\mathrm{sh}(N-1)\, \beta}{\mathrm{sh}\, \beta} \tag{6.13}$$

in which $\mathsf{t}(-\beta) = \mathsf{t}(\beta)$.

Substituting (6.9) into (6.3), we find

$$B(\lambda_{\mathrm{p}}, \lambda_{\mathrm{q}}) = \frac{1}{4} \mathrm{P} \int_{-\infty}^{\infty} \frac{d\beta}{\beta \,\mathrm{sh}\, N\beta} \mathsf{s}(\beta)\, \mathsf{f}(\lambda_{\mathrm{p}}, \beta)\, \mathsf{f}(\lambda_{\mathrm{q}}, -\beta) \tag{6.14}$$

where

$$\mathsf{f}(\lambda_{\mathrm{q}}, \beta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{1 + \lambda_{\mathrm{q}} \mathrm{e}^{i\theta}}{1 - \lambda_{\mathrm{q}} \mathrm{e}^{i\theta}} \, \Delta(\theta)^{i\beta N/\pi} = -\mathsf{f}(\lambda_{\mathrm{q}}^{-1}, \beta) \tag{6.15}$$

We can substitute (6.13) into (6.1) and find

$$A(\lambda_{\mathrm{q}}, t_{\mathrm{p}}) = \frac{1}{4} \ln(-t_{\mathrm{p}} y_{\mathrm{q}}^2)^{N(N-1)} - \frac{1}{4} \mathrm{P} \int_{-\infty}^{\infty} \frac{d\beta t_{\mathrm{p}}^{i\beta N/\pi}}{\beta \,\mathrm{sh}\, N\beta} [\mathsf{t}(\beta) + \mathsf{s}(\beta)]\, \mathsf{f}(\lambda_{\mathrm{q}}, -\beta) \tag{6.16}$$

where (3.18) is used to carry out the integral. If λ_{q} is replaced by $\lambda_{\mathrm{q}}^{-1}$ in the above equation, we can see from (3.18) that we not only need change $\lambda_{\mathrm{q}} \to \lambda_{\mathrm{q}}^{-1}$ in $\mathsf{f}(\lambda, t)$, but also change $y_{\mathrm{q}}^2 \to x_{\mathrm{q}}^2$. In addition, we may use the fact that $\mathsf{s}(\beta)$ is an odd function in β and $\mathsf{t}(\beta)$ is even, to change the integration variable from $\beta \to -\beta$. From (6.13) and (6.2), we obtain

$$C(\lambda_{\mathrm{p}}^{-1}, t_{\mathrm{q}}) = \frac{1}{4} \ln(-t_{\mathrm{q}} x_{\mathrm{p}}^2)^{N(N-1)} - \frac{1}{4} \mathrm{P} \int_{-\infty}^{\infty} \frac{d\beta t_{\mathrm{q}}^{-i\beta N/\pi}}{\beta \,\mathrm{sh}\, N\beta} [\mathsf{t}(\beta) + \mathsf{s}(\beta)]\, \mathsf{f}(\lambda_{\mathrm{p}}^{-1}, \beta) \tag{6.17}$$

We now restrict ourselves to the regimes $-\pi/N \leqslant \arg(t_{\mathrm{p}}), \arg(t_{\mathrm{q}}) \leqslant \pi/N$ such that (6.8) and (6.12) hold. Using (6.16) and (6.14) in (4.24), we find that for $|\lambda_{\mathrm{p}}|, |\lambda_{\mathrm{q}}| < 1$ the "dimensionless" (or normalized) free energy can be written in the alternative form

$$N \ln \tilde{\kappa}_{\mathrm{pq}} = \frac{1}{4} (N-1) \ln(J_{\mathrm{q}}/J_{\mathrm{p}}) - \frac{1}{4} \mathrm{P} \int_{-\infty}^{\infty} \frac{d\beta}{\beta \,\mathrm{sh}\, N\beta} [\mathsf{t}(\beta)\, \mathsf{F}_{\mathrm{pq}} - \mathsf{s}(\beta)\, \mathsf{H}_{\mathrm{pq}}] \tag{6.18}$$

where

$$J_{\mathrm{q}} = -\lambda_{\mathrm{q}}^2 x_{\mathrm{q}}^N / y_{\mathrm{q}}^N \tag{6.19}$$

$$F_{\mathrm{pq}} = [\mathsf{f}(\lambda_{\mathrm{p}}, \beta)\, t_{\mathrm{q}}^{-i\beta N/\pi} - \mathsf{f}(\lambda_{\mathrm{q}}, -\beta)\, t_{\mathrm{p}}^{i\beta N/\pi}] \tag{6.20}$$

$$H_{\mathrm{pq}} = [\mathsf{f}(\lambda_{\mathrm{p}}, \beta)\, t_{\mathrm{q}}^{-i\beta N/\pi} + \mathsf{f}(\lambda_{\mathrm{q}}, -\beta)\, t_{\mathrm{p}}^{i\beta N/\pi} - \mathsf{f}(\lambda_{\mathrm{p}}, \beta)\, \mathsf{f}(\lambda_{\mathrm{q}}, -\beta)] \tag{6.21}$$

The integral in (6.15) can be written in terms of the variable $\lambda = \mathrm{e}^{i\theta}$ as

$$\mathsf{f}(\lambda_{\mathrm{q}}, \beta) = \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{d\lambda}{\lambda} \frac{\lambda_{\mathrm{q}}^{-1} - \lambda_{\mathrm{q}}}{(\lambda_{\mathrm{q}}^{-1} + \lambda_{\mathrm{q}} - \lambda^{-1} - \lambda)} \hat{\Delta}(\lambda)^{i\beta N/\pi} \tag{6.22}$$

Now we change the variable from λ to $\xi = \hat{\Delta}(\lambda)^N$, such that

$$\lambda + 1/\lambda = (1 + \mathrm{k}'^2 - \mathrm{k}^2\xi)/\mathrm{k}' = 2\mu, \qquad \lambda = \mu + \sqrt{\mu^2 - 1}$$

$$\frac{d\lambda}{\lambda} = \frac{d\xi}{\lambda} \frac{d\mu}{d\xi} \frac{d\lambda}{d\mu} = -\frac{\mathrm{k}^2}{2\mathrm{k}'} \frac{d\xi}{\sqrt{\mu^2 - 1}} = -\frac{d\xi}{\sqrt{(\xi_0 - \xi)(\xi_0^{-1} - \xi)}} \tag{6.23}$$

The integration over λ around the unit circle is now changed to an integration along a contour $\mathscr{C}_\xi$ around a cut from $\xi_0 = \eta^N$ to $1/\xi_0 = \eta^{-N}$ in the complex ξ-plane. Consequently, we use (6.23) and (1.2) to find

$$\mathsf{f}(\lambda_{\mathrm{q}}, \beta) = -\frac{\mathrm{k}'}{2\pi i \mathrm{k}^2} \oint_{\mathscr{C}_\xi} \frac{d\xi\, \xi^{i\beta/\pi}}{\sqrt{(\xi_0 - \xi)(\xi_0^{-1} - \xi)}} \frac{\lambda_{\mathrm{q}}^{-1} - \lambda_{\mathrm{q}}}{\xi - t_{\mathrm{q}}^N} \tag{6.24}$$

Since $\xi^{i\beta/\pi}$ has a cut from $-\infty$ to 0 with $\xi = |\xi|\, \mathrm{e}^{i\pi}$ above the cut and $\xi = |\xi|\, \mathrm{e}^{-i\pi}$ below the cut, we may, by subtracting the pole contribution at $\xi = t_{\mathrm{q}}^N$, deform the above integration contour. We obtain a new closed contour which has four pieces. The first piece is an integration from $-\infty$ to 0 above the cut. The next piece goes around the branch point 0 on a circle with infinitesimally small radius to a point below the cut. Thirdly we integrate from 0 to $-\infty$ below the cut. Finally, we close the contour of integration by moving around a large circle with infinite radius counterclockwisely. The contributions from the two circles are zero. This yields

$$\mathsf{f}(\lambda_{\mathrm{q}}, \beta) = t_{\mathrm{q}}^{iN\beta/\pi} \left[\frac{\lambda_{\mathrm{q}}^{-1} - \lambda_{\mathrm{q}}}{|\lambda_{\mathrm{q}}^{-1} - \lambda_{\mathrm{q}}|} - \mathrm{sh}\, \beta G_{\mathrm{q}}(\beta) \right] \tag{6.25}$$

where

$$t_{\mathrm{q}}^{iN\beta/\pi} G_{\mathrm{q}}(\beta) = \frac{\mathrm{k}'}{\pi i \mathrm{k}^2} \int_0^\infty \frac{dz\, z^{i\beta/\pi}}{\sqrt{(\xi_0 + z)(\xi_0^{-1} + z)}} \frac{\lambda_{\mathrm{q}}^{-1} - \lambda_{\mathrm{q}}}{z + t_{\mathrm{q}}^N} \tag{6.26}$$

and we have also used the identity

$$(\xi_0 - t_q^N)(\xi_0^{-1} - t_q^N)\,\mathrm{k}^4 = \mathrm{k}'^2(\lambda_q^{-1} - \lambda_q)^2 \tag{6.27}$$

If we let $z = \mathrm{e}^{2x}$ in (6.26), and use the convention[10]

$$t_q^N = -\mathrm{e}^{2iu_q}, \qquad \mathrm{k}'(\lambda_q^{-1} - \lambda_q) = -2\mathrm{k}\mathrm{e}^{iu_q}\cos v_q \tag{6.28}$$

it is straightforward to show that (6.26) is identical to Eq. (27) of ref. 10.

Substituting (6.25) into (6.20) and (6.21), we may express the free energy for $|\lambda_p|, |\lambda_q| < 1$ given by (6.18) in the form

$$N \ln \tilde{\kappa}_{pq} = \frac{1}{4}(N-1)\ln(J_q/J_p) + \frac{1}{4}\,\mathrm{P}\int_{-\infty}^{\infty} d\beta\, \frac{(t_p/t_q)^{i\beta N/\pi}}{\beta\,\mathrm{sh}\,N\beta}\, E_{pq}(\beta) \tag{6.29}$$

where

$$\begin{aligned} E_{pq}(\beta) = {} & N\,\mathrm{sh}(N-1)\,\beta[G_p(\beta) + G_q(-\beta)] \\ & + [N\,\mathrm{sh}\,\beta\,\mathrm{ch}(N-1)\,\beta - \mathrm{sh}\,N\beta][(\mathrm{sh}\,\beta)^{-2} + G_p(\beta)\,G_q(-\beta)] \end{aligned} \tag{6.30}$$

which is identical to (28) and (29) of ref. 10.

Similarly, from (4.25), (6.14), (6.16) and (6.25), we find that the free energy for $|\lambda_p|, |\lambda_q| > 1$ is

$$N \ln \tilde{\kappa}_{pq} = \frac{1}{4}(N-1)\ln(J_p/J_q) + \frac{1}{4}\,\mathrm{P}\int_{-\infty}^{\infty} d\beta\, \frac{(t_p/t_q)^{i\beta N/\pi}}{\beta\,\mathrm{sh}\,N\beta}\, \bar{E}_{pq}(\beta) \tag{6.31}$$

where

$$\begin{aligned} \bar{E}_{pq}(\beta) = {} & -N\,\mathrm{sh}(N-1)\,\beta[G_p(\beta) + G_q(-\beta)] \\ & + [N\,\mathrm{sh}\,\beta\,\mathrm{ch}(N-1)\,\beta - \mathrm{sh}\,N\beta][(\mathrm{sh}\,\beta)^{-2} + G_p(\beta)\,G_q(-\beta)] \end{aligned} \tag{6.32}$$

which is different from (6.30) in the sign of the first term.

Substituting (6.14), (6.16), (6.17) and (6.25) in (4.27), we find that the free energy for $|\lambda_p| > 1$ and $|\lambda_q| < 1$ is given by

$$\begin{aligned} N \ln \tilde{\kappa}_{pq} = {} & \frac{1}{4}(N-1)\ln(J_p/J_q) + \frac{1}{2}N(N-1)[-\pi i + \ln(t_p t_q)] \\ & - 2\sum_{j=1}^{N-1}(N-j)\ln(\omega^{-j/2}t_q - \omega^{j/2}t_p) \\ & + \frac{1}{4}\,\mathrm{P}\int_{-\infty}^{\infty} d\beta\, \frac{(t_p/t_q)^{i\beta N/\pi}}{\beta\,\mathrm{sh}\,N\beta}\,\{\bar{E}_{pq}(\beta) - 2[\mathsf{t}(\beta) + \mathsf{s}(\beta)]\} \end{aligned} \tag{6.33}$$

Now we can use (6.13) to simplify the above equation to

$$N \ln \tilde{\kappa}_{\rm pq} = \frac{1}{4}(N-1)\ln(J_{\rm p}/J_{\rm q}) + \frac{1}{4}\,{\rm P}\int_{-\infty}^{\infty} d\beta\,\frac{(t_{\rm p}/t_{\rm q})^{i\beta N/\pi}}{\beta\,{\rm sh}\,N\beta}\,\bar{E}_{\rm pq}(\beta) \qquad (6.34)$$

which is identical to (6.31). Likewise from (4.26), we find using (6.14), (6.16), (6.17) and (6.25) that the free energy for $|\lambda_{\rm p}|<1$ and $|\lambda_{\rm q}|>1$ is also given by (6.29). Thus, when $-\pi/N \leqslant \arg(t_{\rm p}), \arg(t_{\rm q}) \leqslant \pi/N$, equation (6.29) is valid for $|\lambda_{\rm p}|<1$ and (6.31) valid for $|\lambda_{\rm p}|>1$.

CONCLUSIONS

There are many other questions remaining in the chiral Potts model that need further investigation. For instance, there are many ways of writing the integrals, and it is not clear, which way would be the most efficient for numerical evaluation. It is also rather puzzling why the inversion relation (3.51) does not hold when $|\lambda_{\rm p}|$ and $|\lambda_{\rm q}|$ are on different Riemann sheets in the complex t, or $\xi = t^N$ plane, which has a two sheeted structure.

Are there other solvable models parametrized by higher genus curves? This is no easy question. Exactly solvable models are few and hard to find, because the integrability conditions are rather stringent. For example, the star-triangle equation for the chiral Potts model involves N^3 identities that must be satisfied by at most $3(N-1)$ variables. The integrable chiral Potts model weights (1.1) were originally found not through logical deduction but by guessing.[(1)] Computers were then only used to verify these conjectures, but played no role in the subsequent construction of the proof.

Whenever a calculation is highly repetitive and tedious, the computer is a marvelous tool. However, computers have not—nor will ever have—the intuition or inspiration to do what Baxter did for the chiral Potts model. Therefore, we feel that it is not unreasonable to conclude that the human mind is far superior to the supercomputer, when used properly.

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